INTERNATIONAL MATHEMATICS
TOURNAMENT OF TOWNS

JUNIOR PAPER: YEARS 8,9,10

Tournament 42, Northern Autumn 2020 (A Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. We say that a circle intersects a quadrilateral *properly* if it intersects each of the quadrilateral's sides at two distinct interior points. Is it true that for each convex quadrilateral there exists a circle which intersects it *properly*? (4 points)

2. A pair of distinct positive integers is called *nice* if their arithmetic mean and their geometric mean are both integers. Is it true that for each nice pair there is another nice pair with the same arithmetic mean?

   (The pairs (a, b) and (b, a) are considered to be the same.) (7 points)

3. Alice and Bob are playing the following game. On each turn Alice suggests an integer number and Bob writes down either that number or the sum of that number plus all previously written numbers. Is it always possible for Alice to ensure that at some moment among the written numbers there are

   (a) at least a hundred copies of number 5? (3 points)

   (b) at least a hundred copies of number 10? (4 points)

4. The "cross" pentomino consists of five 1×1 squares where four squares are all adjacent to the fifth one. Is it possible to cut 9 such pentominoes from an 8×8 chessboard, not necessarily cutting along grid lines?
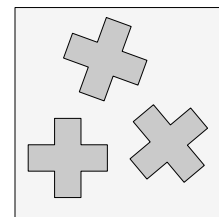
   (The picture shows how to cut three such X pentominoes.) (7 points)

5. Do there exist 100 positive distinct integers such that a cube of one of them equals the sum of the cubes of all the others? (8 points)

6. There are two round tables with n dwarves sitting at each table. Each dwarf has only two friends: his neighbours to the left and to the right. A kind wizard wants to seat the dwarves at one round table so that every two neighbours are friends. His magic allows him to make any $2n$ pairs of dwarves into pairs of friends (the dwarves in a pair may be from the same or from different tables). However, he knows that an evil sorcerer will break n of those new friendships. For which values of n can the kind wizard achieve his goal no matter what the evil sorcerer does?

(10 points)

7. A convex quadrilateral $ABCD$ is such that no three of its sides can form a triangle. Prove that

(a) one of its angles is not greater than $60°$; (6 points)

(b) one of its angles is at least $120°$. (6 points)